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# OPTIMUM CONTROL IN THE PROBLEM OF MINIMIZATION OF HARMFUL IMPURITIES IN THE ATMOSPHERE BY PONTRYAGIN'S MAXIMUM PRINCIPLE AND SPHERICAL HARMONICS METHOD

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Protection of the environment from the industrial pollution is one of the most actual problems of modern science and engineering. This paper is devoted to the investigation of the problem, related to the disposition of industrial objects, which provides the minimal pollution of nearby economically important objects. It is supposed that all of the industrial objects in the given region throw out respective quantities of the harmful impurity in the atmosphere.

*Key words:* The problem of minimization; Integral-differential equation; Optimality Conditions; Principle of the maximum; Harmonics method equations.

## 1. Statement of the problem

Consider the area G of *n*–dimensional space $R^n$ with a border Γ, which has a form of cylinder with bases $\Gamma_0, \Gamma_H$ and lateral surface $\Gamma_l$. We assume that *r* industrial objects are located in the points

$x^i=(x_1^i,x_2^i,...,x_n^i)$, i = 1,2, …, r of G, and throw out $p_i(t)$, $(i=1,2,...,r)$ harmful impurities in the atmosphere. As a result, we come to the following problem setting [1].

It is given the integral-differential equation of the pollution matter diffusion of the *r* industrial objects,

$$\frac{\partial\psi}{\partial t}+\vec{v}grad\psi+\sigma\psi(t,x,\vec{\upsilon})-\eta\sum_{i=1}^{n-1}\frac{\partial^2\psi}{\partial x_i^2}-\xi\frac{\partial^2\psi}{\partial x_n^2}= \\ =\sum_{i=1}^{r}p_i(t)\delta(x-x^i)\delta(\vec{\upsilon}-\vec{\upsilon}^i)+\frac{\lambda}{m(\Omega)}\int_\Omega\Theta(t,x,\vec{\upsilon},\vec{\upsilon}')\psi(t,x,\vec{\upsilon}')d\Omega'. \quad (1)$$

Here, $\psi(t,x,\vec{\upsilon})$ is a concentration of the impurity particles located in the point $x=(x_1,x_2,...,x_n)$ at the moment $t$ and having a velocity $\vec{v}=(v_1,v_2,...,v_n)$, $grad\psi=(\frac{\partial\psi}{\partial x_1},\frac{\partial\psi}{\partial x_2},...,\frac{\partial\psi}{\partial x_n})$ is a vector-gradient, $\vec{v}=(v_1,v_2,...,v_n)\in\Omega-$ is a velocity vector, satisfying to the continuity condition $div(\vec{\upsilon})=\sum_{i=1}^{n}\frac{\partial v_i}{\partial x_i}=0$, and $v_n=0$ at $x_n=0$ and $x_n=H$, that is on $\Gamma_0$ and $\Gamma_H$ ($\Gamma_0$ and $\Gamma_H$ are the bases of the *n* – dimensional cylinder G), Ω is a sphere of unit radius in $R_n$ described by the equation $\sum_{i=1}^{n}v_i^2=1$, *σ*, *λ* are the positive constants describing the medium $G\subset R^n$ where harmful impurities diffuse, $\eta,\xi$ are the coefficients of a "horizontal" and "vertical" turbulent exchange. $x=(x_1,x_2,...,x_n)$ is a spatial point of the area *G*, $\Theta(t,x,\bar{v},\vec{v}')$ is the function describing dispersion of the harmful impurity particles, $\delta(x-x_i),\delta(\bar{v}-\bar{v}_i)$ are the Dirac's *δ*-functions, $m(\Omega)$ is the area of the surface of a unit sphere Ω in $R^n$ [2]:

$$m(\Omega)=\frac{2(\sqrt{\pi})^n}{\Gamma(n/2)},\quad \Gamma(\xi)=\int_0^\infty e^{-t}t^{\xi-1}dt.$$

The non-stationary integro-differential equation (1) must be supplemented with the boundary conditions

$$\psi(t,x,\vec{\upsilon})\big|_{t\le0}=\psi_0(x,\vec{\upsilon}),\qquad (\frac{\partial\psi}{\partial x_n}-\alpha\psi)\big|_{\Gamma_0\times\Omega}=0, \quad (2)$$

$$\frac{\partial\psi}{\partial x_n}\big|_{\Gamma_H\times\Omega}=0,\quad \psi(t,x,\vec{\upsilon})\big|_{\Gamma_\delta\times\Omega}=0\quad at\quad(\vec{\upsilon},\vec{n})<0, \quad (3)$$

where $\vec{n}$ is a normal unit vector to the external side of surface Γ of the cylinder G.

Factor *α* in the condition (2), in the case of three-dimensional space $R^3$, characterizes a probability of the substances, laid-down to the ground surface, to get back into the atmosphere. Condition (3), in the case of *n* = 3, means that the particles which leave the domain *G*, do not return back into the this area.

The problem is to find such functions $p_i(t)$, $(i=1,2,...,r)$, on which the functional

$$J[p]=\sum_{i=1}^{r}\beta_i\int_0^T p_i^2(t)dt+\int_G dG\int_\Omega\left[\psi(T,x,\vec{\upsilon})-\psi_i(x,\vec{\upsilon})\right]^2d\Omega \quad (4)$$

reach the least possible value. Here $\psi(t,x,\vec{\upsilon})$ is the solution of the problem (1)–(3), T > 0 is defined, $\psi_1(x,\vec{\upsilon})$ is the known function from $W_2^{1,0}[G\times\Omega]$, $\beta_i=const>0,(i=1,2,...,r)$.

Admissible controls are the various functions $p=(p_1,p_2,...,p_r)$ from $L_r^2[0,T]$. The control $p=(p_1,p_2,...,p_r)$, which gives the solution of the considered problem, will be called the optimal and denoted by $p^0=(p_1^0,p_2^0,...,p_r^0)$.

**2. Optimality Conditions**

To determine the optimality conditions, we give some admissible increment Δ p = (Δ $p_1$, Δ $p_2$, …, Δ $p_r$) of the control p and denote by $\Delta\psi$ the corresponding increment of the function $\psi(t,x,v)$. It is obvious that the function $\Delta\psi(t,x,v)$ is the solution of the boundary-value problem $[2]$

$$\frac{\partial\Delta\psi}{\partial t}+\vec{v}grad\Delta\psi+\sigma\Delta\psi(t,x,\vec{\upsilon})-\eta\nabla^2\Delta\psi-\xi\frac{\partial^2\Delta\psi}{\partial z^2}= \\ =\sum_{i=1}^{r}\Delta p_i(t)\delta(x-x_i)\delta(\vec{\upsilon}-\vec{\upsilon}_i)+\frac{\lambda}{m(\Omega)}\int_{\Omega}\Theta(t,x,\vec{\upsilon},\vec{\upsilon}')\Delta\psi(t,x,\vec{\upsilon})d\Omega. \tag{5}$$

$$\Delta\psi\big|_{t\le 0}=0,\quad \left(\frac{\partial\Delta\psi}{\partial x_n}-\alpha\Delta\psi\right)\Big|_{\Gamma_0\times\Omega}=0,\ \Delta\psi\big|_{\Gamma\delta\times\Omega}=0\quad at\quad (\vec{\upsilon},\vec{n})<0,\quad \frac{\partial\Delta\psi}{\partial x_n}\Big|_{\Gamma_H\times\Omega}=0 \tag{6}$$

By the direct calculations we find that the functional $J[p]$ (see (1.4)) has the increment

$$\Delta J[p]=\sum_{i=1}^{r}\beta_i\left[\int_0^T 2p_i(t)\Delta p_i(t)dt+\int_0^T\left[\Delta p_i(t)\right]^2dt\right]+ \\ +2\int_G dG\int_{\Omega}\left[\psi(T,x,\vec{\upsilon})-\psi_i(x,\vec{\upsilon})\right]\Delta\psi(T,x,\vec{\upsilon})d\Omega+\int_G dG\int_{\Omega}\left[\Delta\psi(T,x,\vec{\upsilon})\right]^2d\Omega.. \tag{7}$$

Let's consider the arbitrary function $\Phi(t,x,v)\in W_2^{0,1,0}$. Then, obviously that the next equality takes a place,

$$\int_0^T dt\int_G dG\int_{\Omega}\Phi(t,x,\vec{\upsilon})\left\{\frac{\partial\psi}{\partial t}+\vec{v}grad\psi+\sigma\psi-\eta\sum_{i=1}^{n-1}\frac{\partial^2\psi}{\partial x_i^2}-\xi\frac{\partial^2\psi}{\partial x_n^2}-\right. \\ \left.-\sum_{i=1}^{r}p_i(t)\delta(x-x^i)\delta(\vec{\upsilon}-\vec{\upsilon}^i)-\frac{\lambda}{m(\Omega)}\int_{\Omega}\Theta(t,x,\vec{\upsilon},\vec{\upsilon}')\psi(t,x,\vec{\upsilon}')d\Omega'\right\}d\Omega=0.$$

Denoting the left hand side of this equality by $A[\Phi,p]$, we obtain

$$\Delta A[\Phi,p]=\int_0^T dt\int_G dG\int_{\Omega}\Phi(t,x,\vec{\upsilon})\left\{\frac{\partial\psi}{\partial t}+\vec{v}grad\psi+\sigma\psi-\eta\sum_{i=1}^{n-1}\frac{\partial^2\psi}{\partial x_i^2}-\xi\frac{\partial^2\psi}{\partial x_n^2}-\right. \\ \left.-\sum_{i=1}^{r}p_i(t)\delta(x-x^i)\delta(\vec{\upsilon}-\vec{\upsilon}^i)-\frac{\lambda}{m(\Omega)}\int_{\Omega}\Theta(t,x,\vec{\upsilon},\vec{\upsilon}')\psi(t,x,\vec{\upsilon}')d\Omega'\right\}d\Omega=0. \tag{8}$$

Integrating by parts, we transform the equality (2.4) to the form of

$$\int_g dG\int_{\Omega}\Delta\psi(T,x,\vec{\upsilon})\Phi(T,x,\vec{\upsilon})d\Omega+\int_0^T dt\int_G dG\int_{\Omega}\Delta\psi(t,x,\vec{\upsilon})\times \\ \times\left\{-\frac{\partial\Phi}{\partial t}-\vec{v}grad\Phi+\sigma\Phi-\eta\sum_{i=1}^{n-1}\frac{\partial^2\Phi}{\partial x_i^2}-\xi\frac{\partial^2\Phi}{\partial x_n^2}-\frac{\lambda}{m(\Omega)}\int_{\Omega}\Theta(t,x,\vec{\upsilon},\vec{\upsilon}')\Phi(t,x,\vec{\upsilon}')d\Omega'\right\}d\Omega- \\ -\sum_{i=1}^{r}\int_0^T\Delta p_i(t)\Phi(t,x_i,\vec{\upsilon}_i)dt+\int_0^T dt\int_{\Gamma}d\Gamma\int_{\Omega}\left[\vec{\upsilon}_{\vec{n}}\Delta\psi\Phi-\eta\left(\Phi\frac{\partial\Delta\psi}{\partial\vec{n}}-\Delta\psi\frac{\partial\Phi}{\partial\vec{n}}\right)\right]d\Omega+ \\ +\int_0^T dt\int_{\Omega}d\Omega\left[\int_{\Gamma_0}\xi\left(\Phi\frac{\partial\Delta\psi}{\partial x_n^2}-\Delta\psi\frac{\partial\Phi}{\partial x_n^2}\right)-\int_{\Gamma_H}\xi\left(\Phi\frac{\partial\Delta\psi}{\partial x_n^2}-\Delta\psi\frac{\partial\Phi}{\partial x_n^2}\right)\right]d\Gamma=0. \tag{9}$$

Here $\vec{v}_{\vec{n}}$ is a projection of the vector $\vec{v}$ to the unit vector $\vec{n}$.

Up to now $\Phi(t,x,\bar{v})$ was the arbitrary function from $W_2^{0,1,0}([0,T]\times G\times\Omega)$. Let's define it now as a generalized solution of the boundary-value problem

$$\frac{\partial\Phi}{\partial t}+\vec{v}grad\Phi-\sigma\Phi(t,x,\vec{\upsilon})+\eta\sum_{i=1}^{n-1}\frac{\partial^2\Phi}{\partial x_i^2}+\xi\frac{\partial^2\Phi}{\partial x_n^2}+\frac{\lambda}{m(\Omega)}\int_\Omega\Theta(t,x,\vec{\upsilon}',\vec{\upsilon})\Phi(t,x,\vec{\upsilon}')d\Omega'=0 \quad (10)$$

$$\left(\frac{\partial\Phi}{\partial x_n}-\alpha\Phi\right)\Big|_{\Gamma_0\times\Omega}=0,\quad \frac{\partial\Phi}{\partial x_n}\Big|_{\Gamma_H\times\Omega}=0,\quad \Phi\Big|_{\Gamma_0\times\Omega}=0\quad at\quad (\vec{\upsilon},\vec{n})\geq 0$$

$$\Phi(T,x,\vec{\upsilon})=-2[\psi(T,x,\vec{\upsilon})-\psi_1(x,\vec{\upsilon})]. \quad (11)$$

Taking into account (5), (6), (10), and (11), the equality (9) can be simplified. Namely, the second term at the left side in (9) vanishes due to (10). Because of the third equality from conditions (6) and the second of the conditions (11), and since $\cos(\vec{n},x_1)=\cos(\vec{n},x_2)=...=\cos(\vec{n},x_{n-1})=0$ and $\vec{v}_{\vec{n}}=0$ on $\Gamma_0$ and $\Gamma_H$,

$$\int_0^T dt\int_\Gamma d\Gamma\int_\Omega \vec{\upsilon}_n\Delta\psi\Phi d\Omega=0.$$

By virtue of the second condition in (6) and first of the conditions in (11), we have

$$\int_0^T dt\int_{\Gamma_O} d\Gamma\int_\Omega \nu\left(\Phi\frac{\partial\Delta\psi}{\partial x_n}-\Delta\psi\frac{\partial\Phi}{\partial x_n}\right)d\Omega=0.$$

In view of the last condition from (6) and a penultimate condition from (11), the equation (9) takes the form of

$$2\int_G dG\int_\Omega\Delta\psi(T,x,\vec{\upsilon})\left[\psi(T,x,\vec{\upsilon})-\psi_1(x^i,\vec{\upsilon}^i)\right]d\Omega+\sum_{j=1}^{r}\int_0^T\Delta p_j(t)\Phi(t,x^i,\vec{\upsilon}^j)dt=0.$$

From this, by virtue of the last condition from (11), it follows that the increment $\Delta J[p]$ of the minimized functional from (7) is transformed to the form of

$$\Delta J[p]=\sum_{i=1}^{r}\int_0^T\Delta p_i(t)\left[2\beta_i p_i(t)-\Phi(t,x^i,\vec{\upsilon}^i)\right]dt+$$
$$+\sum_{i=1}^{r}\beta_i\int_0^T[\Delta p_i(t)]^2dt+\int_G dG\int_\Omega[\Delta\psi(T,x,\vec{\upsilon}]^2d\Omega. \quad (12)$$

Now, applying a technique of the work [3], the following theorem can be proved.

**Theorem (**principle of the maximum**)**. Necessary and enough condition of optimality of the admissible control $p^0=(p_1^0,...,p_r^0)$ and corresponding to it solution of the boundary-value problem (1)–(3) is a satisfying by the functions

$$H_i(\Phi(t,x^i,\upsilon^i),\psi,p_i)=p_i\Phi(t,x^i,\upsilon^i)-\beta_i p_i^2,\quad i=1,\dots,r \quad (13)$$

of the conditions

$$H_i(\Phi_i^0,\psi^0,p_i^0)=\max H_i(\Phi_i^o,\psi^0,P_i),\quad i=1,\dots,r, \quad (14)$$

where $\Phi_i^0=\Phi^0(t,x^i,v^i),(i=1,2,...,r)$ is the solution to the boundary-value problem (10)–(11) subject to $\psi=\psi_0$.

**3. Construction of optimal control**

For the optimal control construction, first we assume, that no restrictions are imposed on the domain of admissible control parameters. Then it follows from (13), (14) that optimal control $p^0=(p_1^0,...,p_r^0)$ must satisfy the conditions

$$p_i(t)=\frac{1}{2\beta_i}\Phi(t,x^i,\vec{\upsilon}^i),\quad i=1,...,r \tag{15}$$

Thus, the problem of construction of optimal control is reduced to the determining of $p^0=\left(p_1^0,...,p_r^0\right)$, $\psi^0$ and $\Phi^0$ from the equations (1)–(3), (5), (6) and (10), (11).

For the simplicity of reasoning henceforward, we assume that $n$ = 3 and then $x_1=x, x_2=y, x_3=z$, and the unit velocity vector in this case is $\vec{v}=(v_1,v_2,v_3)$, where $v_1=\sin\theta\cos\varphi, v_2=\sin\theta\sin\varphi, v_3=\cos\theta$. We will investigate the boundary-value problem (10)–(11), where, in accordance with [3, 4], we assume

$$\Theta(t,x,y,z,\zeta,\varphi)=g(\mu_0),\quad \mu_0=\zeta'\zeta+\sqrt{1-\zeta^2}\sqrt{1-\zeta'^2}\cos(\varphi-\varphi') \tag{16}$$

Then the equations (10) and (1) take the form of

$$\begin{gathered}\frac{\partial\Phi}{\partial t}+\sin\theta\cos\varphi\frac{\partial\Phi}{\partial x}+\sin\theta\sin\varphi\frac{\partial\Phi}{\partial y}+\cos\theta\frac{\partial\Phi}{\partial z}-\eta(\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2})+\xi\frac{\partial^2\Phi}{\partial z^2}+\\+\frac{\lambda}{4\pi}\int_0^{2\pi}d\varphi'\int_{-1}^{1}g(\mu_0)\Phi(t,x,y,z,\zeta',\varphi')d\zeta'=0\end{gathered} \tag{17}$$

$$\begin{gathered}\frac{\partial\psi}{\partial t}+\sin\theta\cos\varphi\frac{\partial\psi}{\partial x}+\sin\theta\sin\varphi\frac{\partial\psi}{\partial y}+\cos\theta\frac{\partial\psi}{\partial z}+\sigma\psi\left(t,x,y,z,\varphi,\theta\right)-\\-\eta(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2})-\xi\frac{\partial^2\psi}{\partial z^2}=\frac{\lambda}{4\pi}\int_0^{2\pi}d\varphi'\int_{-1}^{1}g(\mu_0)\psi(t,x,y,z,\zeta',\varphi')d\zeta'+\\+\frac{1}{2}\sum_{i=1}^{r}\frac{1}{\beta_i}\Phi\left(t,x^i,y^i,z^i,\varphi,\theta\right)\delta\left(x-x^i,y-y^i,z-z^i\right)\delta\left(\varphi-\varphi^i,\theta-\theta^i\right)\end{gathered} \tag{18}$$

We apply the spherical harmonics method to equation (17). For that, we consider the system of spherical functions [4]:

$$C_k^0=P_k^0(\cos\theta),\quad C_k^m=P_k^m(\cos\theta)\cos m\varphi,\quad S_k^m=P_k^m(\cos\theta)\sin m\varphi \tag{19}$$

k = 0, 1, 2...; m = 0, 1, 2, …, k.

Here

$$P_k^0(\mu)=P_k(\mu)=\frac{1}{2^k k!}\frac{d^k}{d\mu^k}\left[(\mu^2-1)^k\right],\quad k=0,1,2,... \tag{20}$$

are Legendre polynomials [5, 6, 7, 8, 9],

$$P_k^m(\mu)=(1-\mu^2)^{m/2}\frac{d^mP_k(\mu)}{d\mu^m}=\frac{(1-\mu^2)^{m/2}}{2^k k!}\frac{d^{k+m}}{d\mu^{k+m}}\left[(\mu^2-1)^k\right],\quad k=0,1,2,...;\quad m=0,1,2,...,k \tag{21}$$

are the attached Legendre polynomials [4, 5, 7–9]. It is known, that functions (20) and (21) satisfy the orthogonally conditions of on the interval [–1, 1],

$$\int_{-1}^{1}P_k^m(\mu)P_j^m(\mu)d\mu=\frac{2}{2k+1}\frac{(k+m)!}{(k-m)!}\delta_k^j,\quad where\quad \delta_k^j=\begin{cases}1, & j=k\\0, & j\neq k\end{cases} \tag{22}$$

Function $g(\mu_0)$ can be presented as (see (16))

$$g(\mu_0)=\frac{1}{2}\sum_{k=0}^{\infty}(2k+1)g_kP_k(\mu_0)\quad where\quad g_k=\int_{-1}^{1}P_k(\mu_0)g(\mu_0)d\mu_0 \tag{23}$$

Here

$$P_k(\mu_0)=P_k(\zeta)P_k(\zeta')+2\sum_{j=1}^{k}\frac{(k-j)!}{(k+j)!}P_k^j(\zeta)P_k^j(\zeta')\cos(\varphi-\varphi') \tag{24}$$

Solution of the equation (17) will be found in the form of

$$\Phi = \frac{1}{2\pi}\left\{\sum_{k=0}^{\infty}\sum_{m=0}^{k}\frac{2k+1}{1+\delta_m^0}\frac{(k-m)!}{(k+m)!}C_k^m A_k^m + \sum_{k=1}^{\infty}\sum_{m=1}^{k}(2k+1)\frac{(k-m)!}{(k-m)!}S_k^m B_k^m\right\} \tag{25}$$

where $C_k^m$, $S_k^m$ are determined by formulas (19)–(21), and $A_k^m$, $B_k^m$ are unknown functions of arguments *t, x, y, z*.

The system of spherical functions (19) forms the orthogonal functions on the unit sphere and complete function set in the Hilbert space. Therefore any continuous function $\Phi(t,x,y,z,\varphi,\theta)$ can be decomposed on the spherical functions to any accuracy. In the decomposition (25), coefficients are defined by means of the integrals

$$A_k^0 = \int_0^{2\pi} d\varphi \int_{-1}^{1} P_k^0(\mu)\Phi d\mu, \qquad A_k^m = \int_0^{2\pi} d\varphi \int_{-1}^{1} P_k^m(\mu)\cos m\varphi \Phi d\mu,$$
$$B_k^m = \int_0^{2\pi} d\varphi \int_{-1}^{1} P_k^m(\mu)\sin m\varphi \Phi d\mu \tag{26}$$

For convenience, we present function (25) as

$$\Phi = \frac{1}{4\pi}\left\{\sum_{k=0}^{\infty}(2k+1)P_k^0(\zeta)A_k^0 + 2\sum_{k=1}^{\infty}(2k+1)\sum_{m=1}^{k}\frac{(k-m)!}{(k+m)!}P_k^m(\zeta)(A_k^m\cos m\varphi + B_k^m \sin m\varphi)\right\}$$

Using this function and equalities (22)–(24), integral term in the equation (17) can be transformed to

$$J = \frac{\lambda}{4\pi}\int_0^{2\pi} d\varphi' \int_{-1}^{1} g(\mu_0)\Phi(t,x,y,z,\varphi',\zeta')d\zeta' = \frac{\lambda}{8\pi}\left\{\sum_{i=0}^{\infty}(2i+1)g_i P_i^0(\mu)A_i^0 + \right.$$
$$\left. +2\sum_{i=1}^{\infty}(2i+1)g_i\sum_{j=1}^{i}\frac{(i-j)!}{(i+j)!}P_i^j(\mu)(A_i^j\cos j\varphi + B_i^j \sin j\varphi)\right\} \tag{27}$$

Now, the equation (17) can be presented as

$$\frac{\partial\Phi}{\partial t} + \sqrt{1-\mu^2}\cos\varphi\frac{\partial\Phi}{\partial x} + \sqrt{1-\mu^2}\sin\varphi\frac{\partial\Phi}{\partial y} + \mu\theta\frac{\partial\Phi}{\partial z} - \sigma\Phi + \eta\left(\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2}\right) + \xi\frac{\partial^2\Phi}{\partial z^2} +$$
$$+\frac{\lambda}{8\pi}\left\{\sum_{i=0}^{\infty}(2i+1)g_i P_i^0(\mu)A_i^0 + 2\sum_{i=1}^{\infty}(2i+1)g_i\sum_{j=1}^{i}\frac{(i-j)!}{(i+j)!}P_i^j(\mu)(A_i^j\cos j\varphi + B_i^j\sin j\varphi)\right\} = 0 \tag{28}$$

Equation (28) can be reduced to the system of differential equations with respect to $A_k^m, B_k^m, (k=0,1,2,\ldots,\ m=0,1,\ldots,k)$. For that, we multiply the equation (28) in turn by $P_k^0(\mu), (k=0,1,2,\ldots), C_k^m = P_k^m(\mu)\cos m\varphi$, and $S_k^m = P_k^m(\mu)\sin m\varphi$ $(k=0,1,2,\ldots,m=0,1,\ldots,k)$, and integrate with respect to angular variables *φ* and *μ* in the limits from 0 up to 2*π* and from –1 up to 1, respectively. The following recurrence relation from [4, 5] are used:

$$(2k+1)\mu P_k^m(\mu) = (k-m+1)P_{k+1}^m(\mu) + (k+m)P_{k-1}^m(\mu),$$
$$\sqrt{1-\mu^2}P_k^m(\mu) = \frac{1}{2k+1}\left[P_{k+1}^{m+1}(\mu) - P_{k-1}^{m+1}(\mu)\right],$$
$$\sqrt{1-\mu^2}P_k^m(\mu) = \frac{1}{2k+1}\left[(k+m)(k+m-1)P_{k-1}^{m-1}(\mu) - (k-m+1)(k-m+2)P_{k+1}^{m-1}(\mu)\right], \tag{29}$$
$$0 \le m \le k-1$$

So, we multiply (28) by $P_k(\mu) = P_k^0(\mu)$ and integrate a result with respect to variables *φ* and *μ*. Then, by virtue of the first of the formulas (26), we get

$$\int_0^{2\pi} d\varphi \int_{-1}^{1}\frac{\partial\Phi}{\partial t}P_k^0(\mu)d\mu = \frac{\partial}{\partial t}\int_0^{2\pi} d\varphi\int_{-1}^{1}\Phi P_k^0(\mu)d\mu = \frac{\partial A_k^0}{\partial t} \tag{30}$$

Thus, we have found a transformation of the first term in the equation (28).

The second term of this equation will be equal to

$$\int_0^{2\pi} d\varphi \int_{-1}^{1} \sqrt{1-\mu^2}\cos\varphi \frac{\partial\Phi}{\partial x} P_k^0(\mu)\,d\mu = \frac{1}{2k+1}\frac{\partial}{\partial x}\left(A_{k+1}^1 + A_{k-1}^1\right) \tag{31}$$

Here, we used the first two formulae in (26), and the second identity from (29):

$$\int_0^{2\pi} d\varphi \int_{-1}^{1} \sqrt{1-\mu^2} P_k^0(\mu)\cos\varphi \frac{\partial\Phi}{\partial x} d\mu = \frac{\partial}{\partial x}\int_0^{2\pi} d\varphi \int_{-1}^{1} \frac{1}{2k+1}\left[P_{k+1}^1(\mu) - P_{k-1}^1(\mu)\right]\cos\varphi\Phi\,d\mu =$$

$$= \frac{1}{2k+1}\frac{\partial}{\partial x}\left[\int_0^{2\pi} d\varphi \int_{-1}^{1} P_{k+1}^1(\mu)\cos\varphi\Phi\,d\mu - \int_0^{2\pi} d\varphi \int_{-1}^{1} P_{k-1}^1(\mu)\cos\varphi\Phi\,d\mu\right] = \frac{1}{2k+1}\frac{\partial}{\partial x}\left(A_{k+1}^1 + A_{k-1}^1\right)$$

The third term is treated by the similar way,

$$\int_0^{2\pi} d\varphi \int_{-1}^{1} \sqrt{1-\mu^2} P_k^0(\mu)\sin\varphi \frac{\partial\Phi}{\partial y} d\mu = \frac{1}{2k+1}\frac{\partial}{\partial y}\int_0^{2\pi} d\varphi \int_{-1}^{1} \left[P_{k+1}^1(\mu) - P_{k-1}^1(\mu)\right]\sin\varphi\Phi\,d\mu =$$

$$= \frac{1}{2k+1}\frac{\partial}{\partial y}\left[\int_0^{2\pi} d\varphi \int_{-1}^{1} P_{k+1}^1(\mu)\sin\varphi\Phi\,d\mu - \int_0^{2\pi} d\varphi \int_{-1}^{1} P_{k-1}^1(\mu)\sin\varphi\Phi\,d\mu\right]$$

As a result, we have

$$\int_0^{2\pi} d\varphi \int_{-1}^{1} \sqrt{1-\mu^2} P_k^0(\mu)\sin\varphi \frac{\partial\Phi}{\partial y} d\mu = \frac{1}{2k+1}\frac{\partial}{\partial y}\left(B_{k+1}^1 - B_{k-1}^1\right)$$

Now, we consider the fourth term of that equation. We use the first identity from (28) for *m*=0:

$$\int_0^{2\pi} d\varphi \int_{-1}^{1} \mu P_k^0(\mu)\frac{\partial\Phi}{\partial z} d\mu = \frac{1}{2k+1}\int_0^{2\pi} d\varphi \int_{-1}^{1} \frac{\partial\Phi}{\partial z}\left[(k+1)P_{k+1}^0(\mu) + kP_{k-1}^0(\mu)\right]d\mu =$$

$$= \frac{1}{2k+1}\frac{\partial}{\partial z}\left\{(k+1)\int_0^{2\pi} d\varphi \int_{-1}^{1} \Phi P_{k+1}^0(\mu)\,d\mu + k\int_0^{2\pi} d\varphi \int_{-1}^{1} \Phi P_{k-1}^0(\mu)\,d\mu\right\} = \frac{1}{2k+1}\frac{\partial}{\partial z}\left\{(k+1)A_{k+1}^0 + kA_{k-1}^0\right\} \tag{32}$$

The fifth, sixth, seventh and eighth terms contain constant coefficients, therefore they are transformed to the following expression,

$$-\sigma A_л^0 + \eta\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)A_k^0 + \xi\frac{\partial^2 A_k^0}{\partial z^2} \tag{33}$$

Now we consider the next term,

$$\frac{\lambda}{8\pi}\int_0^{2\pi} d\varphi \int_{-1}^{1} P_k^0(\mu)\sum_{i=0}^{\infty}(2i+1)g_i P_i^0(\mu)A_i^0\,d\mu = \frac{\lambda}{8\pi}2\pi\sum_{i=0}^{\infty}(2i+1)g_i A_i^0\int_{-1}^{1} P_k^0(\mu)P_i^0(\mu)\,d\mu =$$

$$= \frac{\lambda}{4}\sum_{i=0}^{\infty}(2i+1)g_i A_i^0\frac{2}{2k+1}\delta_k^i = \frac{\lambda}{4}(2k+1)g_k A_k^0\frac{2}{2k+1} = \frac{\lambda}{2}g_k A_k^0 \qquad (k = 0,1,2,...)$$

Combining the formulae (29)–(33) and $\frac{\lambda}{2}g_k A_k^0$, we obtain the system with respect to $A_k^0$, $A_k^1$, $B_k^1$,

$$\frac{\partial A_k^0}{\partial t} + \frac{1}{2k+1}\left\{\frac{\partial}{\partial x}\left(A_{k+1}^1 - A_{k-1}^1\right) + \frac{\partial}{\partial y}\left(B_{k+1}^1 - B_{k-1}^1\right) + \frac{\partial}{\partial x}\left((k+1)A_{k+1}^0 + kA_{k-1}^0\right)\right\} +$$

$$+\left(\frac{\lambda}{2}g_k - \sigma\right)A_k^0 + \eta\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)A_k^0 + \xi\frac{\partial^2}{\partial z^2}A_k^0 = 0, \qquad k = 0,1,2,... \tag{34}$$

Let's multiply now equation (28) by $C_k^m = P_k^m(\mu)\cos m\varphi$ for $(k = 0, 1, 2, \ldots, m = 0, 1, \ldots, k)$ and integrate the result with respect to $\varphi$ and $\mu$ from 0 to $2\pi$ and from –1 to 1, respectively. As a result, the first term takes a form of $\frac{\partial}{\partial t} A_k^m$. To find expression for the second term, we use the second and third of identities (28).

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