МАТЕМАТИКА

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ASYMPTOTIC OF THE SOLUTION OF A CHEMICAL REACTION PROBLEM WITH STATIONARY REACHABILITY

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Abstarct: *Here it is construct an asymptotic behavior of the solution of a chemical reaction with stationary reachability at the end of the reaction.*
Key words: *Model equation, Cauchy problem, singular point, asymptotic.*

ХИМИЯЛЫК РЕАКЦИЯНЫН МАСЕЛЕСИНИН ЧЕЧИМИНИН СТАЦИОНАРДЫК АБАЛГА ЖЕТИШИНИН АСИМПТОТИКАСЫ

Аннотация :*Мында химиялык реакциянын маселесинин чечиминин стационардык абалга жетишүүсүнүн асимтотикасы изилденет. Чечимдин эки зоналуу асимтотикасы тургузулду.*
Ачкыч сөздөр: *Моделдик теңдеме, Коши маселеси, өзгөчө чекит, асимптотика.*

АСИМПТОТИКА РЕШЕНИЯ ЗАДАЧИ ХИМИЧЕСКОЙ РЕАКЦИИ СО СТАЦИОНАРНОЙ ДОСТИЖИМОСТЬЮ

Аннотация *Здесь строиться асимптотика решения химической реакции со стационарной достижимостью в конце реакции. Построена дву зонная асимптотика решения задачи.*
Ключевые слова*. Модельное уравнение, задача Коши, особая точка , асимтотика.*

1. Introduction

The chemical problem is described [1] by the following Cauchy problem for the differential equation

$$\frac{dT}{dt}=\frac{\varepsilon}{\beta}\left(1+\beta-T\right)e^{\frac{T-1}{\varepsilon T}}, \qquad (1)$$

$$T\left(0\right)=1. \qquad (2)$$

Its exact solution is given by the formula

$$t(T)=\frac{\beta}{\varepsilon}\left\{e^{-\frac{1}{\varepsilon}}Ei\left(\frac{1}{T\varepsilon}\right)-e^{-\frac{1}{\varepsilon}+\frac{1}{\varepsilon(1+\beta)}}Ei\left(\frac{1}{\varepsilon T}-\frac{1}{\varepsilon(1+\beta)}\right)\right\}-$$

$$\frac{\beta}{\varepsilon}\left\{e^{-\frac{1}{\varepsilon}}Ei\left(\frac{1}{\varepsilon}\right)-e^{-\frac{1}{\varepsilon}+\frac{1}{\varepsilon(1+\beta)}}Ei\left(\frac{1}{\varepsilon}-\frac{1}{\varepsilon(1+\beta)}\right)\right\}.$$

here is the sign of P.V. means the integral is understood in the main meaning of Cauchy.

Obtaining the asymptotic behavior of the solution to problem (1) - (2) from the exact solution is a difficult task. In [1], the zero asymptotic of the solution of this problem in three steps was obtained.

2. Construction an external solution

This solution we will seek in the form

$$T=1+\varepsilon T_1+\varepsilon^2 T_2+...+\varepsilon^n T_n+..., \quad (3)$$

here $T_i=T_i(t)(i=1,2,...)$ is while unknown functions.

Substituting series (3) in (1), after the usual procedures, we obtain the following asymptotics for determining unknown functions:

$$T=1+\varepsilon ln\frac{1}{1-t}+\varepsilon^2\frac{1}{1-t}\left\{a_0+\frac{\varepsilon}{1-t}a_1+...+\left(\frac{\varepsilon}{1-t}\right)^n a_n+...\right\},\varepsilon\to 0, \quad (4)$$

here $a_n=const$.

Series (4) is asymptotic only on the interval $[0,1-\varepsilon^{\alpha}],\ (0<\alpha<1)$. At the point t = 1 the asymptotic property is lost.

Therefore, in a neighborhood of the point t = 1, we introduce the extended variable σ by follow formula

$$1-t=e^{-\sigma/\varepsilon}, 0\le\sigma<1$$

Then problem (1) *reduce* to the form:

$$\frac{du(\sigma)}{d\sigma}=\frac{1}{\beta}(1+\beta-u(\sigma))e^{\frac{1}{\varepsilon}\left(1-\sigma-\frac{1}{u(\sigma)}\right)} \quad (5)$$

here $u(\sigma)=T(1-e^{-\sigma/\varepsilon})$.

3. Construction an external solution(second way)

Definition 2. The variable σ is called an internal variable. The solution of the equation (5) is named internal solution of the problem (1)-(2).

In order to get a limited solution, we require

$$1-\sigma-\frac{1}{u(\sigma)}=O(\varepsilon),\varepsilon\to 0 \Leftrightarrow u(\sigma)\sim\frac{1}{1-\sigma},\ \varepsilon\to 0$$

Therefore, we are looking for solution (5) in the form:

$$u(\sigma)=\frac{1}{1-\sigma}+\varepsilon u_1(\sigma)+\varepsilon^2 u_2(\sigma)+\ldots+\varepsilon^n u_n(\sigma)+\ldots. \quad (6)$$

Substituting (6) into (5), to determine unknown functions, we obtain the following equations:

$$\frac{1}{(1-\sigma)^2}=\frac{1}{\beta}\left(1+\beta-\frac{1}{1-\sigma}\right)e^{(1-\sigma)^2 u_1}, \quad (7)$$

$$u'_1=\frac{1}{\beta}\left(\left(1+\beta-\frac{1}{1-\sigma}\right)e^{(1-\sigma)^2 u_1}(1-\sigma)^2\left(u_2-u_1^2+u_1^2\sigma\right)-u_1 e^{(1-\sigma)^2 u_1}\right), \quad (8)$$

The solution of equation (7) is representable in the form

$$u_1=\frac{1}{(1-\sigma)^2}ln\frac{\beta}{(1-\sigma)\left((1+\beta)(1-\sigma)-1\right)}=-\frac{1}{(1-\sigma)^2}ln(1-\sigma)\left(1-\frac{1+\beta}{\beta}\sigma\right)=$$

$$=-\frac{1}{(1-\sigma)^2}ln\frac{1+\beta}{\beta}(1-\sigma)-\frac{1}{(1-\sigma)^2}ln\left(\frac{\beta}{1+\beta}-\sigma\right),\ 0\le\sigma<\frac{\beta}{1+\beta},$$

From here we have

$$u_1(\sigma)\sim-(\beta+1)^2\,ln\left(\frac{\beta}{1+\beta}-\sigma\right).$$

It is we have follow equation for $u_2(\sigma)$:

$$u'_1=\left(u_2-u_1^2+u_1^2\sigma\right)-\frac{u_1}{(1-\sigma)\left((1+\beta)(1-\sigma)-1\right)},$$

From here we have got

$$u_2=u'_1+\frac{u_1}{(1-\sigma)\left((1+\beta)(1-\sigma)-1\right)}+u_1^2(1-\sigma),\ \ u_2(0)=\frac{1-2\beta}{\beta},$$

he following estimate is true:

$$u_2 \sim (\beta+1)^2 \frac{1}{\frac{\beta}{1+\beta}-\sigma} ln\left(\frac{\beta}{1+\beta}-\sigma\right), \ \sigma \to \gamma = \beta/_{1+\beta}.$$

Analogously we have

$$u_3 \sim u_2^2 = \left((\beta+1)^2 \frac{1}{\frac{\beta}{1+\beta}-\sigma} ln\left(\frac{\beta}{1+\beta}-\sigma\right) \right)^2, \ \sigma \to \gamma = \beta/_{1+\beta}.$$

Substituting the found asymptotic in (5) we have:

$$u(\sigma) = \frac{1}{1-\sigma} - \varepsilon\frac{1}{(1-\sigma)^2} ln(\gamma-\sigma) - \frac{(1+\beta)\varepsilon^2}{(1-\sigma)^3(\gamma-\sigma)} ln(\gamma-\sigma) -$$

$$-\frac{1}{2}\varepsilon^3(1-\sigma)^{-2}\frac{(1+\beta)^2}{(\gamma-\sigma)^2} ln^2(\gamma-\sigma) + \varepsilon O\left(\left(\varepsilon\frac{ln(\gamma-\sigma)}{\gamma-\sigma}\right)^3\right), \quad (9)$$

here $\gamma = \frac{\beta}{1+\beta}$.

Thus, we have proved the following theorem

Theorem 1. Solutions to problem (1) exist on the interval $\sigma \in [0,\gamma-\varepsilon^{\lambda}], \ (0<\lambda<1)$ and the asymptotic (9) holds for it.

To find the asymptotic of $u(\sigma)$ for $\sigma \to \gamma$, we put in (13)

$$\sigma = \tilde{\sigma} = \gamma - r(\varepsilon), (0 < r(\varepsilon), r(0)=0).$$

Then we have:

$$u(\tilde\sigma) = \frac{1}{1-\tilde\sigma} - \varepsilon\frac{1}{(1-\tilde\sigma)^2} ln(\gamma-\tilde\sigma) - \frac{(1+\beta)\varepsilon^2}{(1-\tilde\sigma)^3(\gamma-\tilde\sigma)} ln(\gamma-\tilde\sigma) -$$

$$-\frac{1}{2}\varepsilon^3(1-\tilde\sigma)^{-2}\frac{(1+\beta)^2}{(\gamma-\tilde\sigma)^2} ln^2(\gamma-)\tilde\sigma + \varepsilon O\left(\left(\varepsilon\frac{ln(\gamma-\tilde\sigma)}{\gamma-\tilde\sigma}\right)^3\right),$$

Thus, we have proved the following theorem

Theorem 1. Solutions to problem (1) exist on the interval

$\sigma \in [0,\gamma-\varepsilon^{\lambda}], \ (0<\lambda<1)$ and the asymptotic (9) holds for it.

To find the asymptotic $u(\sigma)$, for σ→γ, we put in (13)

$$\sigma=\tilde{\sigma}=\gamma-r(\varepsilon),(0<r(\varepsilon),r(0)=0)$$

Then we have:

$$u(\tilde{\sigma})=\frac{1}{1-\tilde{\sigma}}-\varepsilon\frac{1}{(1-\tilde{\sigma})^2}ln(\gamma-\tilde{\sigma})-\frac{(1+\beta)\varepsilon^2}{(1-\tilde{\sigma})^3(\gamma-\tilde{\sigma})}ln(\gamma-\tilde{\sigma})-$$

$$-\frac{1}{2}\varepsilon^3(1-\tilde{\sigma})^{-2}\frac{(1+\beta)^2}{(\gamma-\tilde{\sigma})^2}ln^2(\gamma-)\tilde{\sigma}+\varepsilon O\left(\left(\varepsilon\frac{ln(\gamma-\tilde{\sigma})}{\gamma-\tilde{\sigma}}\right)^3\right),$$

Hence, equating the middle terms to zero, we have

$$r^2(\varepsilon)+(1+\beta)\varepsilon r(\varepsilon)+\frac{\varepsilon^2}{2}(1+\beta)^2 \ln r(\varepsilon)=0.$$

Solving this as a quadratic equation for $r(\varepsilon)$ we have::

$$r(\varepsilon)=-\frac{1+\beta}{2}\varepsilon+\sqrt{\frac{(1+\beta)^2\varepsilon^2}{4}+\frac{\varepsilon^2}{2}(1+\beta)^2 ln\frac{1}{r(\varepsilon)}}$$

or

$$r(\varepsilon)\square-\frac{1+\beta}{2}\varepsilon+\frac{\varepsilon}{\sqrt{2}}(1+\beta)ln\frac{1}{r(\varepsilon)},r(\varepsilon)\to 0.$$

Since

$$(1-\sigma)^{-1}=\frac{1}{1-\gamma}\left(1+\frac{r(\varepsilon)}{1-\gamma}\right)^{-1}\sim(1+\beta)(1+(1+\beta)r(\varepsilon)),(\gamma=\frac{\beta}{1+\beta}).$$

Therefore

$$u(\sigma)=1+\beta-(1+\beta)^2\left(\frac{\varepsilon}{\sqrt{2}}(1+\beta)ln\varepsilon^{-1}\right),\varepsilon\to 0.$$

Since $1-t=e^{-\sigma/\varepsilon}, 0\le\sigma<1,$ the variable t cannot be greater than 1.

4. Construction an internal solution

To construct the asymptotic solution for t> 1, we introduce another new variable s.

If we make a substitution

$$t-1=\frac{\beta}{\varepsilon}e^{-\beta/(1+\beta)\varepsilon}s\sim \begin{array}{l} s=(t-1)\frac{\beta}{\varepsilon}e^{\beta/(1+\beta)\varepsilon}=(t=1-e^{-\sigma/\varepsilon}) \\ =-\frac{\beta}{\varepsilon}e^{(\gamma-\sigma)/\varepsilon} \end{array}$$

Let

$$s_0=-\frac{\beta}{\varepsilon}e^{r(\varepsilon)/\varepsilon},u(\tilde{\sigma})=1+\beta-(1+\beta)^2\left(\frac{\varepsilon}{\sqrt{2}}(1+\beta)ln\varepsilon^{-1}\right),\varepsilon\to 0.$$

We introduce the notation T (t) =$\psi(s)$ then equation (1) in the new variables takes the form:

$$\frac{d\psi}{ds}=(1+\beta-\psi)e^{\frac{\psi-(1+\beta)}{\varepsilon\psi(1+\beta)}}\ . \quad (11)$$

Definition 1. The variable *s* is called an internal variable. The solution of the equation (11) is named internal solution.

Note that at the point s_0

$$\psi(s_0)=1+\beta-O\left(\varepsilon\left(ln\frac{1}{\varepsilon}\right)^{-1}\right):=k\,.$$

The asymptotic solution of equation (10) we seek in the form:

$$\psi(s)=1+\beta+\varepsilon(1+\beta)^2\psi_1(s)+\varepsilon^2(1+\beta)^2\psi_2(s)+\dots, \quad (11)$$

Substituting (14) into (13) for unknown functions, we obtain the following problems

$$\psi'_1(s)=-\psi_1(s)e^{\psi_1(s)}\ ,\ \psi_1(s_0)=k\ , \qquad (12.1)$$

$$\psi'_2(s)=-\psi_2(s)e^{\psi_1(s)}-\psi_1(s)\big(\psi_2(s)-\psi_1(s)(1+\beta)\big)e^{\psi_1(s)},\psi_2(s_0)=0, \qquad (12.2)$$

$$\begin{aligned}&\psi'_3(s)=-\psi_3(s)e^{\psi_1(s)}-\psi_2(s)\big(\psi_2(s)-\psi_1(s)(1+\beta)\big)e^{\psi_1(s)}-\\&-\psi_1(s)e^{\psi_1(s)}\Big(\psi_3(s)-2(1+\beta)\psi_1(s)\psi_2(s)+\psi_1^3(s)+\frac{1}{2}\psi_2^2(s)\\&+(1+\beta)^2+-(1+\beta)\psi_2(s)\psi_1^2(s)+\frac{1}{2}(1+\beta)^2\psi_1^4(s)\Big),\psi_3(s_0)=0\end{aligned} \qquad (12.3)$$

The solution of equation (15.1) has the form

$$\int_{-k}^{-u} \frac{e^{\tau}}{\tau} d\tau = -s + s_0, \ (u = \psi_1) \qquad (13)$$

The solution of equation (15.1) has the form

$$\int_{-u_0}^{-u} \frac{e^{-\tau} - 1}{\tau} d\tau + ln(-u) - ln u_0 = -s + s_0$$

or

$$\int_{-u_0}^{-u} \frac{e^{-\tau} - 1}{\tau} d\tau + O(u) + ln(-u) - ln u_0 = -s + s_0 .$$

From here, we get:

$$u = \psi_1 = -e^{-s+s_0} + O\left(e^{-s+s_0}\right), \ s \to \infty, (t > 1).$$

In this way,

$$\psi_1(s) = -e^{s_0 - s} + O\left(e^{-s+s_0}\right), s \to \infty (t > 1).$$

Now we solve the problem (15.2)

$$M\psi_2(s) := \psi'_2(s) + \left(1 - \psi_1(s)\right) e^{\psi_1(s)} \psi_2(s) = -\psi_1^2(s) e^{\psi_1(s)}, \ \psi_1(u_0) = 0 \quad (14)$$

Homogeneous equation (14) has a solution

$$V(s) = \psi_2^{одн}(s) = \psi_1'(s) = e^{-s+s_0} + O\left(e^{-s+s_0}\right), t > 1.$$

Given this, from (17) we have:

$$\psi_2(s) = \int_{s_0}^{s} V(s) V^{-1}(\rho) u^2(\rho) e^{u(s)} ds = u^2(s), s \to +\infty, t > 1$$

etc.

$$\psi_k(s) = u^k(s), s \to +\infty.$$

Therefore, we got that

$$\psi(s) = 1 + \beta + \varepsilon(1+\beta)^2 e^{-s+s_0} + \left(\varepsilon(1+\beta)e^{-s}\right)^2 + ... + O\left(\varepsilon(1+\beta)e^{-s}\right)^n + ..., \ s \to \infty, \varepsilon \to 0$$

.

Comment. Thus, the solution to this problem begins a jump at a singular point

$$\tilde{t} = 1 - e^{\sigma_0/\varepsilon}, \ \sigma_0 = \frac{\beta}{1+\beta} - \varepsilon ln \frac{1}{\varepsilon} + O\left(\varepsilon ln \frac{1}{\varepsilon}\right)^2,$$

and

$$T\left(\tilde{t}\right)=1+\beta-\varepsilon ln\frac{1}{\varepsilon}+0\left(\varepsilon ln\frac{1}{\varepsilon}\right),\varepsilon\to 0.$$

Then it will quickly exponentially move to the equilibrium point T=1+β.

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