УДК: 515.122

КИНЕМАТИКАЛЫК МЕЙКИНДИКТЕРДИН ӨЛЧӨМДӨРҮ ЖАНА КОМПЬЮТЕРДЕ КӨРСӨТҮҮСҮ
КОМПЬЮТЕРНОЕ ПРЕДСТАВЛЕНИЕ И РАЗМЕРНОСТИ КИНЕМАТИЧЕСКИХ ПРОСТРАНСТВ
COMPUTER PRESENTATION AND DIMENSIONS OF KINEMATICAL SPACES

P.S. Pankov, Institute of Mathematics
720071 Bishkek, Kyrgyzstan E-mail: pps5050@mail.ru
A.H. Zhoraev docent of Department of Mechanics and Mathematics
Kyrgyz-Uzbek University
79 Isanov Str. 714017 Osh, Kyrgyzstan
E-mail: zhvl967@mail.ru

Аннотация: *Компьютер аркылуу жүзөгө ашырылуучу, эвклиддик эмес топологиялык мейкиндиктерде башкарылуучу кыймылдоо бул макалада каралат. Макалада чекиттин жана узун-туурасы бар объекттин кыймылдоосун жабдуучу аныктамалар жана кинематикалык мейкиндиктерде кыймылдоонун негизинде өлчөмдүү үч аныктама бар.*

Аннотация. *В статье рассматривается управляемое движение в неэвклидовых топологических пространствах, которое может быть реализовано на компьютере. Статьясодержит определения, обеспечивающие движение точки и протяженных объектов, и три определения размерности, основанные на движении в кинематических пространствах.*

Abstract: *This paper deals with controlled motion in non-Euclidean topological spaces which can be implemented by means of computer. It contains a survey of definitions to provide motion of a point, definitions of motion of a lengthy object and three definitions of dimension based on motion in kinematical spaces.*

Key words: *topological space, metrical space, kinematical space, computer, Riemann surface, motion, rotation, dimension.*

Урунттуу сөздөр: *топологиялык мейкиндик, метрикалык мейкиндик, кинематикалык мейкиндик, компьютер, римандык бет, кыймылдоо, айлануу, өлчөм.*

Ключевые слова: *топологическое пространство, метрическое пространство, кинематическое пространство, компьютер, риманова поверхность, движение, вращение, размерность.*

1. Introduction

Since it is known, S.Ulam [6] was the first to propose an active work on computer to present a virtual (four-dimensional Euclidean) space, but he did not propose any concrete methods of implementation.

An another way to perform non-Euclidean spaces visually by means of computer was proposed [7]. His idea can be demonstrated by the following example. If we put the figure $\supseteq$ onto a common ring band and we can look "along" the band sufficiently far then we will see the se- quince of diminishing figures $\supseteq\supseteq\supseteq\supseteq\supseteq$….
Ifwe do same for a Mobius band then we will see the sequence of diminishing figures $\supseteq\subseteq\supseteq\subseteq\supseteq$….

We [4] proposed to use controlled (interactive) motion in non-Euclidean topological spa- ces by means of computer. We implemented the Mobius band as follows. We are standing on a band and see the figure $\supseteq$ (the horizon is less than half of the length of the band). We go and soon we see the figure $\subseteq$.

We [1] introduced general conception of a kinematical space and implemented some ki- nematical spaces (Riemann surfaces, Mobius band, projective plane, topological torus) with se- arch in them. Methods of constructing such spaces and marking to facilitate motion in them were proposed in [2] and applied in [5].

A similar definition, independently of us, was proposed in [3]. We do not know whether it was implemented by computer.

Kinematical investigation of unknown spaces defined by differential and algebraic equa- tions was proposed in [8].

New types of dimensions based on motion were announced in [9] and [10].

In this paper we expound this approach and give definitions of three new types of dimen- sions: successful observation and "almost observation" from observable domains; possibility of rotation of lengthy sets.

2. Review of preceding definitions on motion and dimensions

We will use denotations $R := (-\infty,\infty)$; $R_+ := [0, \infty)$; $Q^k := [0; 1]^k$, $k = 1, 2, 3,...$ is a k-dimen- sional cube (segment, square, cube, ...); ε is a small positive parameter. Also, we will extend func- tions to sets with same denotations.

Natural motion of points (also implemented on computer) is presented by the following sys- tem of axioms [2] based on the notion of *time*.

Definition 1. A computer program is said to be a **presentation** of a computer kinematical space if:

P1) there is an(infinite) metrical space X of points and a set X_1 of program-presentable points being sufficiently dense in X;

P2)the user can pass from any point x_1 in X_1 to any other point x_2 by a sequence of adjacent points in X_1 by their will;

P3) the minimal time to reach x_2 from x_1 is (approximately) equal of the minimal time to reach x_2 from x_1.

The space X is said to be a **kinematic space**;the space X_1 is said to be a **computer kinematic space**;this minimal time is said to be the **kinematical distance** ρ_X between x_1 and x_2; a sequence of adjacent points is said to be a **route**. Passing to a limit as X_1 tends to X we obtain the following.

There is a set K of **routes**;each route M, in turn, consists of the positive real number T_M (**time** of route) and the function $m_M: [0, T_M] \to X$ (**trajectory** of route);

(K1) For $x_1 \neq x_2 \in X$ there exists such $M \in K$ that $m_M(0) = x_1$ and $m_M(T_M) = x_2$, and the set of values of such T_M is bounded with a positive number below;

(K2=P3) If $M=\{T_M , m_M(t)\} \in K$ then the pair $\{T_M , m_M(T_M - t)\}$ is also a route of K (the reverse motion with same speed is possible);

(K3) If $M=\{T_M, m_M(t)\} \in K$ and $T^* \in (0, T_M)$ then the pair: T^* and function $m^*(t)=m_M(t)$ $(0 \leq t \leq T^*)$ is also a route of K (one can stop at any desired moment);

(K4) concatenation of routes for three distinct points x_1, x_2, x_3.

Remark 1. After our publication [2] another version of presenting "motion" based on the notion of "path" was proposed.

Denote the set of connected subsets of R as In. A *path* is a continuous map $\gamma : In \to X$ (a topological space).

Definition 2. The following definition is composed of some definitions in [3] (briefly) redu- ced to a "a priori" bounded, path-connected space X; denotations are slightly unified.

A length structure in X consists of a class A of admissible paths together with a function (length) $L: A \to R_+$.

The class A has to satisfy the following assumptions:

(A1) The class A is closed under restrictions: if $\gamma \in A$, $\gamma : [a, b] \to X$ and $[u, v] \subset [a, b]$ then the restriction $\gamma|_{[u, v]} \in A$ and the function L is continuous with respect to u, v;

(A2)The class A is closed under concatenations of paths and the function L is additive correspond- dingly. Namely, if a path γ: $[a, b] \to X$ is such that its restrictions γ_1, γ_2 to $[a, c]$ and $[c, b]$ belong to A, then so is γ.

(A3)The class A is closed under (at least) linear reparameterizations and the function L is

invariant correspondingly: for a path $\gamma \in A$, $\gamma : [a, b] \to X$ and a homeomorphism $\varphi : [c, d] \to [a, b]$ of the form $\varphi(t) = \alpha t + \beta$, the composition $\gamma(\varphi(t))$ is also a path.
(A4) (similar to (Kl)).
The metric in X is defined as
$\rho_L(z_0, z_1) := \inf \{L(\gamma) \mid \gamma : [a, b] \to X;\ \gamma \in A;\ \gamma(a) = z_0;\ \gamma(b) = z_1\}$.

We mention some known definitions briefly (we restrict with metric sets):

Definition 3. Dim-dimension (or "cover"- or Lévesque one): it is defined to be the mini- mum value of n, such that every open cover (set of open sets) C of X has an open refinement with number of overlapping being $(n + 1)$ or below.

Ind-dimension: by induction $Ind(\varnothing) = -1$; $Ind(X)$ is the smallest n such that, for every closed subset F of every open subset U of X, there is an openset V in "between F and U " such that $Ind(Boundary(U)) < (n - 1)$.

Minkovski (Min)-dimension. $Min(X) := \lim\{ (-\log N_\varepsilon / \log \varepsilon) | \to 0\}$ where N_ε is the mini- mal cardinality of ε-sets in X. If $\lim$ does not exist then $\liminf$ $(Min\_)$ and $\limsup(Min_+)$ to be con- sidered.

Remark 3. For metrical spaces Dim-dimension and Ind-dimension coincide. Obviously, $Min(Q^k) = k$.

3.Motion of lengthy objects in kinematical spaces

Definition 1 is not sufficient for motion of point sets. One of possible extensions of Defi- nition 1 is the demand of isometric of all shifts of a set during motion but it is too binding. We proposed [11]

Definition 4. Given a set $S \subset K$. A set of routes with functions $\{M(p) : p \subset S\}$ with a same time T is said to be a motion of S with bounded deformation if there are such constants $0 < a\_ < 1 < a_+$ that
(M1) $(\forall p \in S)(M(p)(0)=p)$;
(M2) $(\forall p_1 \neq p_2 \in S)(\forall t \in [0,T])(\rho_K(M(p_1)(t),M(p_2)(t)) \in [a_-,a_+]\,\rho_K(p_1,p_2))$.

Definition 5. If additionally
(R1) there exists such set ("axis") $C \in S$ that $M|_C$ is the identity operator;
(R2) $(\forall p \in S)\{M(S)(0) = M(S)(T))$ (initial and final sets coincide);
(R3) $(\forall t_1 \neq t_2 \in (0,T))(M(S)(t_1) \cap M(S)(t_2) = C)$ (the set S is "thin" and does not pass by itself excluding the axis);
then such motion is said to be a "proper rotation" (with "bounded deformation" correspondingly) around C.

Remark 4. To define "rotation" of a general (spacious) objects in a space without geo- metry is very complicated. For our purposes such "proper rotation" is sufficient.

4. Dimensions defined by motion in kinematical spaces

Definition 6. A set B of a kinematical space X is said to be "fully observable" if there exists a route including all this set.

Definition 7. A kinematical space X is said to be "locally observable" if each its point has a "fully observable" neighborhood.

Definition 8. A locally observable kinematical space X is said to be "observable" if each its bounded set is "fully observable".

As usually, we will call a bijective continuous image of a segment $[0,T]$ a "segment in kinematical space". Also, we will call the trace of bijective motion of a segment with one of end- points fixed "triangle" etc.

Definition 9. "Orientation dimension" $O\ ri$- is 1 for observable spaces. If there exists such "segment" with endpoints z_1 and z_2 and an inner point z_0 and such rotation with bounded deformation around z_0 that z_1 passes to z_2 and vice versa then $O\ ri\ (K) > 2$; if there exists a "triangle" with vertices z_1, z_2 and z_3 and a point z_0 within the "segment" $z_1 - z_2$ which can be rotated around the segment $z_0 - z_3$ with bounded deformation such that z_1 passes to z_2 and vice versa then $O\ ri\ (K) > 3$ etc.
Obviously, $O\ ri\ (Q^k) = Dim\ (Q^k)$, $k = 1, 2, 3, \ldots$.

Remark 5. "Motion" of such lengthy sets into themselves is not sufficient for such definition because a triangle $z_1 - z_2 - z_3$ can be transformed continuously into triangle $z_2 - z_1 - z_3$ by motion along the Mobius band but its dimension is 2.

The next definition also begins with observable spaces.

Definition 10. (For bounded spaces only).Kinematical *(Kin-)* dimension is 1 for observable spaces. By induction: If *not* $(Kin(X)\leq n), n\geq 1$ and there exists function $M_n(a_1, a_2, ..., a_n, t): R_+^n \times R_+ \to X$ defined for $a_1 \leq a_2 \leq ... \leq a_n$, being a route for fixed $a_1, a_2, ..., a_n$, such that

1) $M_n(a_1, a_2, ..., a_n, 0) = x_0$(a fixed element in K);

2)$M_n(a_1, a_2, ..., a_n, t)$does not depend on a_i being grater than t;

3)$\rho_K(M_n(a_1', a_2', ..., a_n', t), M_n(a_1'', a_2'', ..., a_n'', t)) \leq |a_1' - a_1''| + |a_2' - a_2''| + ... + |a_n' - a_n''|$;

4) Trajectories of $M_n(a_1, a_2, ..., a_n, t)$for all a_i cover the set X

then$Kin(X)=n+1$.

It is obvious that $Kin(Q^1) = 1$.

Remark 6. There exists a continuous Peano surjection $Q^1 \to Q^2$if $Q^2$is considered as a topological space.

Theorem 1.$Kin(Q^2) = 2(= Dim(Q^2))$if $Q^2$is considered as a kinematical space with Euclidean metric.

*Proof.*By contradiction.If $Kin(Q^2) = 1$ then there exists a trajectory S covering all Q^2. Choose a natural n and divide Q^2into $n \times n$little squares. The trajectory S passes though all centers of squares and has the length within each square not less than $1/n$. Hence, its total length is not less than $n \cdot n \cdot 1/n = n$ and tends to infinity as $n \to \infty$.

Definition 11.A bounded kinematical space$X$is said to be "almost observable" if

$(\forall \varepsilon > 0)(\exists M \in K)(\forall x \in X)(\exists t \in [0,T_M])(\rho_X(x, m_M(t)) < \varepsilon)$.

Denote the lower bound of such $T_M$for fixedεas $W_\varepsilon(X)$.

Remark 7. The notion of a compact space can be expressed by "almost observability": if a kinematical space is almost observable and complete then it is compact.

As $N_\varepsilon \approx W_\varepsilon(X)/\varepsilon$we obtain "Minkovski-kinematical" *Min-kin*-dimension:

Definition 12.$Min\text{-}kin\ (X) := 1 - \lim\{ \log W_\varepsilon(X)/\log\varepsilon \mid \varepsilon \to 0\}$. If this *lim*does not exist then*liminf* (*Min-kin_*) and *lim sup*($Min\text{-}kin_+$)to be considered.

For example, $W_\varepsilon(Q^1) = 1 - 2\varepsilon$;

$Min\text{-}kin(Q^1) = 1 - \lim\{\log(l - 2\varepsilon)/\log\varepsilon \mid \varepsilon \to 0\} = 1 - 0 = 1$;

$W_\varepsilon(Q^2) \approx (1 - 2\varepsilon)/(2\varepsilon) + (1 - 2\varepsilon)$; $Min\text{-}kin(Q^2) = 1 + \lim\{\log(2\varepsilon)/\log\varepsilon \mid \varepsilon \to 0\} = 1 + 1 = 2$.

5. Conclusion

The paper demonstrates that various new definitions of "dimension" conforming with known ones can be introduced on the base of "motion" and "rotation" in kinematical spaces.

References:

1. Борубаев А.А., Панков П.С. Компьютерные представления кинематических топологи-ческих пространств. – Бишкек: КГНУ, 1999.
2. Borubaev A.A., Pankov P.S., Chekeev A.A.Spaces Uniformed by Coverings.-Budapest:Hungarian-Kyrgyz Friendship Society, 2003.
3. Burago D., Burago Yu., Ivanov S.A Course in Metric Geometry //Graduate Studies in Mathematics, Volume 33, American Mathematical Society Providence, Rhode Island, 2001.
4. Pankov P.S., Bayachorova B.J. Using computers to perform non-Euclidean topological spaces //The 6-th conference and exhibition on computer graphics and visualization "Graphicon-96", Saint-Petersburg, 1996, vol. 2, p. 232.
5. PankovP.S., JoraevA.H.Recognizability and local computer presentation of topological spaces //Problems of modern topology and applications: abstracts of the international conference. -Tash-kent:Nizami Tashkent State Pedagogical University, 2013. - Pp. 70-72.
6. Ulam S.M.A Collection of Mathematical Problems.- New York: Interscience Publishers, 1960.
7. WeeksJ.R.The Shape of Space.- New York:Marcel Dekker, Inc., 1985.
8. ЖораевА.Х. Исследование топологических пространств кинематическим методом. – Saarbrűcken, Deutschland: Lap Lambert Academic Publishing, 2017. – 78 с.
9. ZhoraevA.H.Motion of sets and orientation dimension of kinematical spaces //Abstracts of the VI Congress of the Turkic World Mathematical Society.- Astana: L.N. Gumilyov Eurasian National

University, 2017. - P. 124.
10. Zhoraev A. Orientation dimension and orientation constants of kinematical spaces //Abstracts of the Third International Scientific Conference "Actual problems of the theory of control, topo- logy and operator equations".- Bishkek:Kyrgyz Mathematical Society, 2017. - P. 36.
11. Жораев А. Х. Индуктивное определение кинематической размерности топологических пространств // Вестник Института математики НАН КР, 2018, № 1. - С. 139-144.