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THE DEFINITION OF THE DISPLACEMENT FUNCTION BY THE ELEMENTS OF CAUCHY'S TENSOR

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Vector field can be defined by its divergence and rotor. This field can be represented as a sum of irrotational field and solenoidal field (the expansion theorem of Helmholtz). The finding of such field leads to the solution of the partial differential equations at some boundary conditions. The other problem appears. For example, the solution of static boundary problem in stress gives the field for stress tensor σ_{ij} *, from which we can define the field of linear Cauchy's deformation tensor* ε_{ij} *. Further more the problem of finding the displacement field, corresponding with the tensor* ε_{ij} *arises. This work is devoted to this problem.*

Let the Cauchy's tensor to be known in the area V with the border S.

$$\varepsilon_{ij}(x_1, x_2, x_3),\ x_i \in V \quad (1),$$

where functions ε_{ij} and their partial derivatives of the first and the second-order are continuous.

Let's define functions u_i on the basis of this tensor. The differential of the functions u_i can be written as

$$du_i = u_{i,j}dx_j = (\varepsilon_{ij} + \omega_{ij})dx_j, \quad (2)$$

where $\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}),\ \omega_{ij} = \frac{1}{2}(u_{i,j} - u_{j,i})$. (3)

It is easy to check that the functions ω_{ij} satisfy the following equations.

$$\omega_{ij,k} = \varepsilon_{ki,j} - \varepsilon_{kj,i} \quad (5)$$

Following Чезаро, we can integrate the expression (2) by the line ℓ, which is lying in the area V

$$u_i(x_1, x_2, x_3) = u_i(x_1^0, x_2^0, x_3^0) + \int_\ell (\varepsilon_{ij} + \omega_{ij})dy_j\,. \quad (6)$$

The equation (6) might be presented in the form of

$$u_i(x_1, x_2, x_3) = u_i(x_1^0, x_2^0, x_3^0) + \int_\ell \varepsilon_{ik}dy_k - \omega_{ij}d(x_j - y_j)$$

Integration in parts reduce this equation to the form

$$u_i(x_1, x_2, x_3) = u_i(x_1^0, x_2^0, x_3^0) + \omega_{ij}(x_1^0, x_2^0, x_3^0)(x_j - x_j^0) + \int_\ell (\varepsilon_{ik} + (x_j - y_j)\omega_{ij,k})dy_k$$

Using the correlation (5), we can present this expression in the form of

$$\begin{aligned} u_i(x_1, x_2, x_3) = u_i(x_1^0, x_2^0, x_3^0) + \omega_{ij}(x_1^0, x_2^0, x_3^0)(x_j - x_j^0) + \\ + \int_\ell (\varepsilon_{ik} + (x_j - y_j)(\varepsilon_{ki,j} - \varepsilon_{kj,i}))dy_k \end{aligned} \quad (7)$$

Here it is not known whether the subintegral expression is a full differential or not. If it's not a full differential, then we can get different values of the function (6), by going to the point (x_1, x_2, x_3) by different lines. For the unambiguity of the function (7) the subintegral expression

$$(\varepsilon_{ik} + (x_j - y_j)(\varepsilon_{ki,j} - \varepsilon_{kj,i}))dy_k \quad (8)$$

must be the full differential.

If the expression (8) is a full differential, than curvilinear integral on any closed line ℓ is equal to zero

$$\oint_\ell (\varepsilon_{ik} + (x_j - y_j)(\varepsilon_{ki,j} - \varepsilon_{kj,i}))dy_k = 0\,.$$

Let the closed line ℓ be a contour of some surface S lying in the area V. Then by the formula of Stokes

$$\oint_{\ell}(\varepsilon_{ik}+(x_j-y_j)(\varepsilon_{ki,j}-\varepsilon_{kj,i}))dy_k=\iint_{s} rot((\varepsilon_{ik}+(x_j-y_j)(\varepsilon_{ki,j}-\varepsilon_{kj,i}))e_k)\cdot\eta ds=0,$$

where e_κ - orts of a rectangular coordinate system, η - Normal to the surface S.
In order to equalize the integral on a surface to zero at any choice of the surface s, it is necessary

$$rot((\varepsilon_{ik}+(x_j-y_j)(\varepsilon_{ki,j}-\varepsilon_{kj,i}))e_k)=0,\ \ x_i\in V$$

This expression we can present as

$$\vartheta_{ijk}((\varepsilon_{pk}+(x_t-y_t)(\varepsilon_{kp,t}-\varepsilon_{kt,p})),_j=0,$$

ϑ_{ijk} - tensor of Levi -Civita.
Let's execute the differentiation

$$\vartheta_{ijk}(\varepsilon_{pk,j}-\delta_{jt}(\varepsilon_{kp,t}-\varepsilon_{kt,p})+(x_t-y_t)(\varepsilon_{kp,tj}-\varepsilon_{kt,pj}))=$$
$$\vartheta_{ijk}(\varepsilon_{pk,j}-(\varepsilon_{kp,t}-\varepsilon_{kt,p})+(x_t-y_t)(\varepsilon_{kp,tj}-\varepsilon_{kt,pj}))=0$$

Because of the symmetry of Cauchy's tensor $(\varepsilon_{ij}=\varepsilon_{ji})$ and the properties of tensor of Levi-Civitas at the rearrangement of the indexes, this expression
$\vartheta_{ijk}\varepsilon_{kj,p}=0,\ \ \varepsilon_{pk,j}-\varepsilon_{kp,j}=0,$
will turn into the following equation

$$(x_t-y_t)\vartheta_{ijk}(\varepsilon_{kp,tj}-\varepsilon_{kt,pj})=0.$$

In order to let this equation to be satisfied at any arbitrary values x_t, y_t, which belong to the area V, it is necessary

$$\vartheta_{ijk}(\varepsilon_{kp,tj}-\varepsilon_{kt,pj})=0 \qquad (9)$$

Here the equation 81 (three free indexes), from which only 6 are independent. These are those equations, which are known as the equations of continuousness (or compatibility of deformation). Let ε_{ij} satisfy the equations (9); then it is possible to write the equation (7) as

$$u_i(x_1,x_2,x_3)=u_i(x_1^0,x_2^0,x_3^0)+\omega_{ij}(x_1^0,x_2^0,x_3^0)(x_j-x_j^0)++\int_{(x_1^0,x_2^0,x_3^0)}^{(x_1,x_2,x_3)}(\varepsilon_{ik}-(x_j-y_j)(\varepsilon_{ki,j}-\varepsilon_{kj,i}))dy_k \qquad (10)$$

Thus, if the subintegral expression (8) satisfies to the equations (9), then the integral in the equation (10) does not depend on the line, connecting points (x_1^0,x_2^0,x_3^0), (x_1,x_2,x_3). It is the easier to take a straight line which is taking place through points (x_1^0,x_2^0,x_3^0), (x_1,x_2,x_3) instead of this line. Entering a designation

$$g_i(x_1,x_2,x_3)=\int_{(x_1^0,x_2^0,x_3^0)}^{(x_1,x_2,x_3)}(\varepsilon_{ik}+(x_j-y_j)(\varepsilon_{ki,j}-\varepsilon_{kj,i}))dy_k, \qquad (11)$$

Let's write the equation (10) as

$$u_i(x_1,x_2,x_3)=u_i(x_1^0,x_2^0,x_3^0)+\omega_{ij}(x_1^0,x_2^0,x_3^0)(x_j-x_j^0)+g_i(x_1,x_2,x_3) \qquad (12)$$

Calculation of the integral (11) manually is rather labor-consuming. System MathCAD successfully carries out such calculation. The program is developed; at first it checks the satisfaction of the equations of compatibility of deformation; then, carrying out the further calculations, defines a vector of displacement $g_i(x_1,x_2,x_3)$, and, consequently, the displacement field u_i. If the displacement field is established, then it is possible to define everything that it creates. In this sense, tensor ε_{ij} is the full characteristic of the intense and deformed conditions.
For example, let the tensor be set

$$\varepsilon(x_1,x_2,x_3)=\begin{pmatrix} 9x_1^8x_2^7-2x_1x_3^2 & \frac{7}{2}x_1^9x_2^6+2x_1^3x_3^7+\frac{1}{2}x_3 & -x_1^2x_3+\frac{5}{2}x_3^4+\frac{5}{2}x_1^4x_2^3x_3 \\ \frac{7}{2}x_1^9x_2^6+2x_1^3x_3^7+\frac{1}{2}x_3 & -4x_2^3x_3^3 & \frac{3}{2}x_1^5x_2^2x_3+2x_2^3x_3^2+\frac{7}{2}x_1^4x_3^6-\frac{3}{2}x_2^4x_3^2+\frac{1}{2}x_1 \\ -x_1^2x_3+\frac{5}{2}x_3^4+\frac{5}{2}x_1^4x_2^3x_3 & \frac{3}{2}x_1^5x_2^2x_3+2x_2^3x_3^2+\frac{7}{2}x_1^4x_3^6-\frac{3}{2}x_2^4x_3^2+\frac{1}{2}x_1 & x_1^5x_2^3+2x_2^4x_3-8x_3^7 \end{pmatrix}$$

The program defines function $g_i(x_1,x_2,x_3)$ as

$$g(x_1, x_2, x_3) = \begin{pmatrix} x_3^{\,5} - x_1^{\,2} x_3^{\,2} + x_1^{\,9} x_2^{\,7} \\ x_1^{\,4} x_3^{\,7} - x_2^{\,4} x_3^{\,3} + x_1 x_3 \\ -x_3^{\,8} + x_2^{\,4} x_3^{\,2} + x_1^{\,5} x_2^{\,3} x_3^{\,5} \end{pmatrix}.$$

The program is added. Those, who are interested in it, can carry out the integration of the right part of the equation (11) with the help of the programme.

The summary

Vector field can be defined by its divergence and rotor. This field can be represented as a sum of irrotational field and solenoidal field (the expansion theorem of Helmholtz). The finding of such field leads to the solution of the partial differential equations at some boundary conditions. The other problem appears. For example, the solution of static boundary problem in stress gives the field for stress tensor σ_{ij}, from which we can define the field of linear Cauchy's deformation tensor ε_{ij}. Further more the problem of finding the displacement field, corresponding with the tensor ε_{ij} arises. This work is devoted to this problem.

In this work the development of Cesaro's formula is set in; also it's shown that condition of independence of its curvilinear integral is congruent to the condition of compatibility of deformation.

Cesaro's formula was almost unused because of the laboriousness of the integration. In this work the program is developed in the system MathCAD, which firstly checks the satisfaction of the equations of compatibility of deformations, and then quite successfully carries out this integration in character form.