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COMPUTER PRESENTATION OF GENERALIZED KINEMATICAL SPACES

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Abstract: *This paper deals with controlled presentation of various metrical and topological spaces which can be implemented by means of computer. The paper contains a survey of preceding methods and definitions to provide presentation of a part of a space and proposes a new generalized definition.*

Key words: *topological space, metrical space, kinematical space, computer, Riemann surface, motion, rotation, dimension.*

ЖАЛПЫЛАНГАН КИНЕМАТИКАЛЫК МЕЙКИНДИКТЕРДИН КОМПЬЮТЕРДЕ КӨРСӨТҮҮСҮ

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Аннотация: *Компьютер аркылуу жүзөгө ашырылуучу, ар түрдүү метрикалык жана топологиялык мейкиндиктердин башкарылуучу көрсөтүүсү бул макалада каралат. Макалада мейкиндиктин бөлүгүнүн мурдагы көрсөтүүсүнүн усулдарын жана аныктамаларын кароо жана узун-туурасы бар объекттин кыймылдоосун жабдуучу аныктамалар жана кинематикалык мейкиндиктерде кыймылдоонун негизинде өлчөмдү үч аныктама бар.*

Урунттуу сөздөр: *топологиялык мейкиндик, метрикалык мейкиндик, кинематикалык мейкиндик, компьютер, римандык бет, кыймылдоо, айлануу, өлчөм.*

КОМПЬЮТЕРНОЕ ПРЕДСТАВЛЕНИЕ ОБОБЩЕННО-КИНЕМАТИЧЕСКИХ ПРОСТРАНСТВ

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Аннотация: *В статье рассматривается управляемое представление различных метрических и топологических пространств, которое может быть реализовано на компьютере. В статье содержится обзор предыдущих методов и определений для представления частей пространства и предлагается новое обобщенное определение.*

Ключевые слова: *топологическое пространство, метрическое пространство, кинематическое пространство, компьютер, риманова поверхность, движение, вращение, размерность.*

1. Introduction

We propose the following: a presentation of a topological space on computer is said to be topological (natural, continuous) if some points of the space are presented and images of close points are close.

We introduce a corresponding definition of space with sets of given lengths which generalizes the known definition of kinematical spaces.

The second section contains a survey of preceding methods and definitions for computer presentation of topological spaces.

2. Survey of preceding results on computer presentation

We will use denotations $R := (-\infty,\infty)$; $R_+ := [0, \infty)$; $Q^k := [0; 1]^k$, $k = 1, 2, 3,...$ is a k-dimen- sional cube (segment, square, cube, ...); ε is a small positive parameter. Also, we will extend func- tions to sets with same denotations.

S.Ulam [6] was the first to propose an active work on computer to present a virtual (four-dimensional Euclidean) space, but he did not propose any concrete methods of implementation.

The idea of [7] can be demonstrated by the following example. If the figure $\supseteq$ is put onto a common ring band and the user can "look along" the band sufficiently far then the user will see the sequence of diminishing figures $\supseteq$ $\supseteq$ $\supseteq$ $\supseteq$ $\supseteq \ldots$.

If the user "does" same for a Mobius band then the user will see the sequence of diminishing figures $\supseteq$ $\subseteq$ $\supseteq$ $\subseteq$ $\supseteq \ldots$.

In [4] it was proposed to use controlled (interactive) motion in non-Euclidean topological spaces by means of computer. For example, the Mobius band was implemented as follows. The user "is standing" on a band and sees the figure $\supseteq$ (the horizon is less than half of the length of the band). The user "goes" and soon see the figure $\subseteq$.

In [1] a general conception of a kinematical space and implemented some kinematical spaces (Riemann surfaces, Mobius band, projective plane, topological torus) with search in them was introduced.

Definition 1. A computer program is said to be a **presentation** of a computer kinematical space if:

P1) there is an (infinite) metrical space X of points and a set X_1 of program-presentable points being sufficiently dense in X;

P2) the user can pass from any point x_1 in X_1 to any other point x_2 by a sequence of adjacent points in X_1 by their will;

P3) the minimal time to reach x_2 from x_1 is (approximately) equal of the minimal time to reach x_2 from x_1.

The space X is said to be a **kinematic space**; the space X_1 is said to be a **computer kinematic space**; this minimal time is said to be the **kinematical distance** ρ_X between x_1 and x_2; a sequence of adjacent points is said to be a **route**. Passing to a limit as X_1 tends to X we obtain the following.

There is a set K of **routes**; each route M, in turn, consists of the positive real number T_M (**time** of route) and the function $m_M : [0, T_M] \rightarrow X$ (**trajectory** of route);

(K1) For $x_1 \neq x_2 \in X$ there exists such $M \in K$ that $m_M(0) = x_1$ and $m_M(T_M) = x_2$, and the set of values of such T_M is bounded with a positive number below;

(K2) If $M= \{T_M , m_M(t)\} \in K$ then the pair $\{T_M , m_M (T_M - t)\}$ is also a route of K (the reverse motion with same speed is possible); (cf. P3).

(K3) If $M= \{T_M, m_M(t)\} \in K$ and $T^* \in (0, T_M)$ then the pair: T^* and function $m^*(t)=m_M(t)$ $(0 \leq t \leq T^*)$ is also a route of K (one can stop at any desired moment);

(K4) concatenation of routes for three distinct points x_1, x_2, x_3.

Methods of constructing such spaces and marking to facilitate motion in them were proposed in [2] and applied in [5].

A similar definition was proposed in [3].

Denote the set of connected subsets of R as In. A *path* is a continuous map $\gamma : In \rightarrow X$ (a topological space).

Definition 2. The following definition is composed of some definitions in [3] (briefly) reduced to a "a priori" bounded, path-connected space X; denotations are slightly unified.
A length structure in X consists of a class A of admissible paths together with a function (length) $L: A \to R_+$.

The class A has to satisfy the following assumptions:
(A1) The class A is closed under restrictions: if $\gamma \in A$, $\gamma : [a, b] \to X$ and $[u, v] \subset [a, b]$ then the restriction $\gamma|_{[u, v]} \in A$ and the function L is continuous with respect to u,v;
(A2) The class A is closed under concatenations of paths and the function L is additive correspond- dingly. Namely, if a path $\gamma : [a, b] \to X$ is such that its restrictions γ_1, γ_2 to $[a, c]$ and $[c, b]$ belong to A, then so is γ.
(A3) The class A is closed under (at least) linear reparameterizations and the function L is invariant correspondingly: for a path $\gamma \in A$, $\gamma : [a, b] \to X$ and a homeomorphism $\varphi : [c, d] \to [a, b]$ of the form $\varphi(t) = \alpha t + \beta$, the composition $\gamma(\varphi(t))$ is also a path.
(A4) (similar to (Kl)).

The metric in X is defined as
$\rho_L(z_0, z_1) := \inf\{L(\gamma) \,|\gamma : [a, b] \to X; \gamma \in A; \gamma(a) = z_0; \gamma(b) = z_1\}$.

Kinematical investigation of unknown spaces defined by differential and algebraic equations was proposed in [8].

Definition 3. Dim-dimension (or "cover"- or Lebesque one): it is defined to be the minimum value of n, such that every open cover (set of open sets) C of X has an open refinement with number of overlappings being $(n + 1)$ or below.

Ind-dimension: by induction $Ind(\emptyset) = -1$; $Ind(X)$ is the smallest n such that, for every closed subset F of every open subset U of X, there is an open set V in "between F and U" such that $Ind(Boundary(U)) < (n — 1)$.

Minkovski (Min)-dimension. $Min(X) := \lim\{ (-\log N_\varepsilon/\log \varepsilon)| \to 0\}$ where N_ε is the minimal cardinality of ε-sets in X. If $\lim$ does not exist then $\lim\inf (Min\_)$ and $\lim\sup (Min_+)$ to be con- sidered.

Remark 3. For metrical spaces *Dim*-dimension and *Ind*-dimension coincide. Obviously, $Min(Q^k) = k$.

New types of dimensions based on motion were announced in [9] and [10].

Definition 1 is not sufficient for motion of point sets. One of possible extensions of Definition 1 is the demand of isometric of all shifts of a set during motion but it is too binding. We proposed [11]

Definition 4. Given a set $S \subset K$. A set of routes with functions $\{M(p) : p \subset S\}$ with a same time T is said to be a motion of S with bounded deformation if there are such constants $0 < a\_ < 1 < a_+$ that
(M1)$(\forall p \in S)(M(p)(0)=p)$;
(M2) $(\forall p_1 \neq p_2 \in S)(\forall t \in [0,T])(\rho_K(M(p_1)(t),M(p_2)(t)) \in [a_-,a_+]\rho_K(p_1,p_2))$.

Definition 5. If additionally
(R1) there exists such set ("axis") $C \in S$ that $M|_C$ is the identity operator;
(R2) $(\forall p \in S)\{M(S)(0) = M(S)(T))$ (initial and final sets coincide);
(R3) $(\forall t_1 \neq t_2 \in (0,T))(M(S)(t_1) \cap M(S)(t_2) = C)$ (the set S is "thin" and does not pass by itself excluding the axis);
then such motion is said to be a "proper rotation" (with "bounded deformation" correspondingly) around C.

Remark 4. To define "rotation" of a general (spacious) objects in a space without geometry is very complicated. For our purposes such "proper rotation" is sufficient.

We proposed

Definition 6. A set B of a kinematical space X is said to be "fully observable" if there exists a route including all this set.

Definition 7. A kinematical space X is said to be "locally observable" if each its point has a "fully observable" neighborhood.

Definition 8. A locally observable kinematical space X is said to be "observable" if each its bounded set is "fully observable".

As usually, we will call a bijective continuous image of a segment $[0,T]$ a "segment in kinematical space". Also, we will call the trace of bijective motion of a segment with one of endpoints fixed "triangle" etc.

Definition 9. "Orientation dimension" *Ori-* is 1 for observable spaces. If there exists such "segment" with endpoints z_1 and z_2 and an inner point z_0 and such rotation with bounded deformation around z_0 that z_1 passes to z_2 and vice versa then $Ori(K) > 2$; if there exists a "triangle" with vertices z_1, z_2 and z_3 and a point z_0 within the "segment" $z_1 - z_2$ which can be rotated around the seg- ment $z_0 - z_3$ with bounded deformation such that z_1 passes to z_2 and vice versa then $Ori(K) >$ 3 etc.

Obviously, $Ori(Q^k) = Dim(Q^k), k = 1, 2, 3, \ldots$.

Remark 5. "Motion" of such lengthy sets into themselves is not sufficient for such definition because a triangle $z_1–z_2–z_3$ can be transformed continuously into triangle $z_2–z_1–z_3$ by motion along the Mobius band but its dimension is 2.

The next definition also begins with observable spaces.

Definition 10. (For bounded spaces only). Kinematical *(Kin-)* dimension is 1 for observable spaces. By induction: If $not(Kin(X)\le n), n \ge 1$ and there exists function $M_n(a_1, a_2, \ldots, a_n, t)$: $R_+^n \times R_+ \to X$ defined for $a_1 \le a_2 \le \ldots \le a_n$, being a route for fixed $a_1, a_2, \ldots, a_n$, such that

1) $M_n(a_1, a_2, \ldots, a_n, 0) = x_0$ (a fixed element in K);
2) $M_n(a_1, a_2, \ldots, a_n, t)$ does not depend on a_i being greater than t;
3) $\rho_K(M_n(a_1', a_2', \ldots, a_n', t), M_n(a_1'', a_2'', \ldots, a_n'', t)) \le | a_1' - a_1''| + |a_2' - a_2''| + \ldots + |a_n' - a_n''|$;
4) Trajectories of $M_n(a_1, a_2, \ldots, a_n, t)$ for all a_i cover the set X

then $Kin(X)=n+1$.

It is obvious that $Kin(Q^l) = 1$.

4. Definition of generalized kinematical spaces

Definition 11. There is a family K of subsets of the set X called **lengthies**; each **lengthy** has the **length** >0**.**

The space X is said to be a **generalized kinematic space**.

(G1) For each $x_1 \ne x_2 \in X$ there exists such lengthy $M \in K$ that $x_1, x_2 \in M$ and the set of lengths of such M is bounded with a positive number below; this infinum is said to be the **generalized kinematical distance** ρ_X between x_1 and x_2.

(G2) If $x_1, x_2 \in M_1$ and $x_2, x_3 \in M_2$ then there exists such lengthy $M_3 \in K$ that $x_1, x_2, x_3 \in M_3$ and $length(M_3) \le length(M_1) + length(M_2)$.

If

(G3) For each $x_1 \ne x_2 \in X$ there exists such lengthy $M_{12} \in K$ that $length(M_{12}) = \rho_X (x_1, x_2)$ then the generalized kinematical space X is said to be **flat** (with respect to K).

If a lengthy is presented as a route then Definition 11 generalizes Definition 1.

In this paper we expound this approach and give definitions new types of dimensions: successful observation and "almost observation" from observable domains.

Definition 12. If X as a set is a lengthy then the generalized kinematic space X is said to be 1-dimensional with respect to K.

Definition 13. A bounded generalized kinematical space X is said to be "almost observable" if

$(\forall\varepsilon > 0)(\exists M \in K)($ *Hausdorff distance between* X *and* $M < \varepsilon)$.

Denote the lower bound of length of such M for fixed ε as $W_\varepsilon(X)$.

The notion of a compact space can be expressed by "almost observability": if a generalized kinematical space is almost observable and complete then it is compact.

As $N_\varepsilon \approx W_\varepsilon(X)/\varepsilon$ we obtain "Minkovski-kinematical" *Min-kin*-dimension:

Definition 14. $Min\text{-}kin(X) := 1 - \lim\{ \log W_\varepsilon(X)/\log \varepsilon \mid \varepsilon \to 0\}$. If this *lim* does not exist then *lim inf* ($Min\text{-}kin_-$) and *lim sup* ($Min\text{-}kin_+$) to be considered.

5. Conclusion

We hope that the new definitions in this paper would provide more effective computer presentations for various types of topological and metric spaces.

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